Spectre de grandes matrices aléatoires perturbées

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Main topic of random matrix theory

We consider a $n \times n$ matrix $X_n = (x_{i,j}^{(n)})$ whose entries are random variables.

The main topic of this field is the study of the eigenvalues and eigenvectors of X_n as $n \to \infty$.

The empirical spectral measure

Let us note $\lambda_1, \ldots, \lambda_n$ the eigenvalues of X_n.
The empirical spectral measure of X_n is the probability measure defined by:

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The empirical spectral measure of X_n is the probability measure defined by:

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For example, in the case of a Hermitian random matrix X_n, for $A \subseteq \mathbb{R}$:

$$\mu_{X_n}(A) = \frac{1}{n} \, \#\big\{\lambda_i \in A \, ; \; i \in \{1, \dots, n\}\big\}$$

Wigner's Semicircle Law (1958)

If $X_n = (x_{i,j}^{(n)})$ is a $n \times n$ real symmetric random matrix such that

1. $\mathbb{E}(x_{i,j}^{(n)}) = 0$ for $1 \leq i \leq j \leq n$
2. $\mathbb{E}(|x_{i,j}^{(n)}|^2) = 1$ for $1 \leq i < j \leq n$
3. for all $k \in \mathbb{N}$, $\sup_{i,j} \mathbb{E}(|x_{i,j}^{(n)}|^k) = C(k) < \infty$

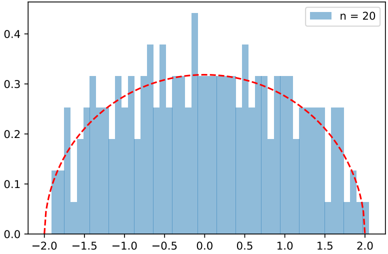
then

$$\mu_{\frac{X_n}{\sqrt{n}}} \quad \xrightarrow[n\to\infty]{\text{dist.}} \quad \mu_{sc}$$

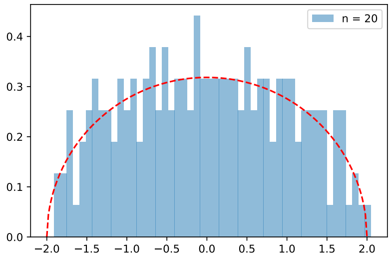
for

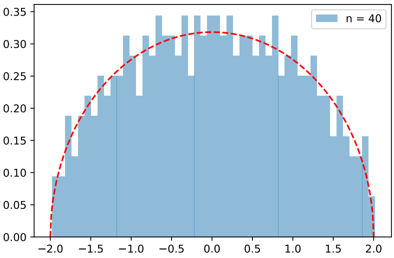
$$d\mu_{sc}(t) = \frac{1}{2\pi}\sqrt{4 - t^2}\ \mathbb{1}_{[-2,2]}(t)\mathrm{d}t$$

Wigner's Semicircle Law



Wigner's Semicircle Law





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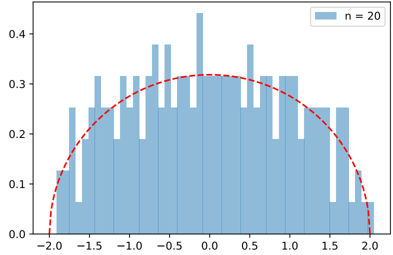

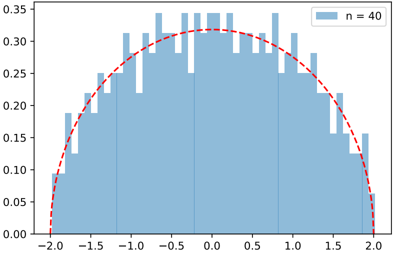

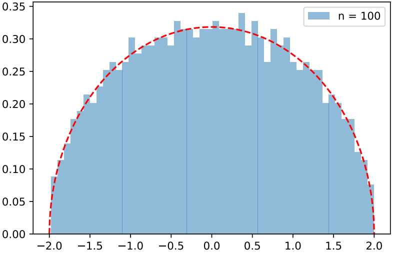

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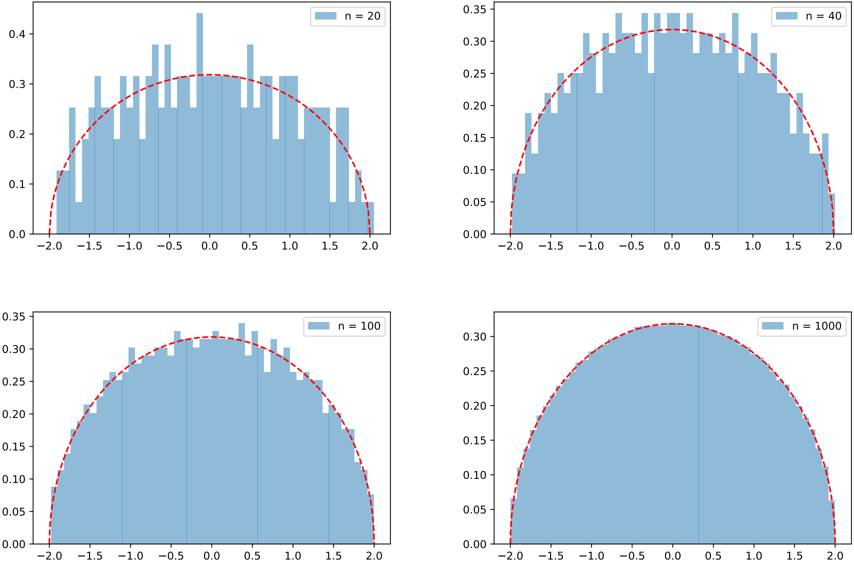

A pertubation problem

How are the spectral properties of an operator altered when the operator is subject to a small perturbation ?

Study of the spectrum of a perturbed matrix

$$H_n$$

- H_n is a deterministic Hermitian matrix.

Study of the spectrum of a perturbed matrix

$$H_n + \quad H_n'$$

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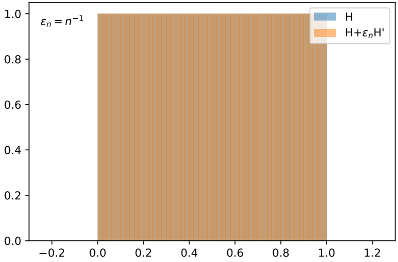
Study of the spectrum of a perturbed matrix

$$H_n + \varepsilon_n . H'_n$$

- H_n is a deterministic Hermitian matrix.
- H'_n is a random Hermitian matrix which operator norm is of order 1.
- (ε_n) is a positive sequence such that $\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$



$\varepsilon_n = n^{-1}$
H
H+ε_nH'
1.0
0.8
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0.4
0.2
0.0
−0.2
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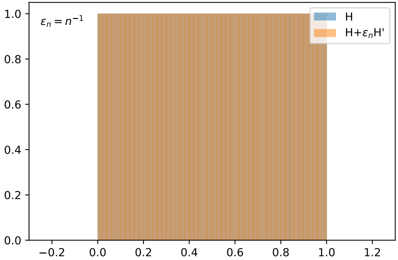
0.2
0.4
0.6
0.8
1.0
1.2



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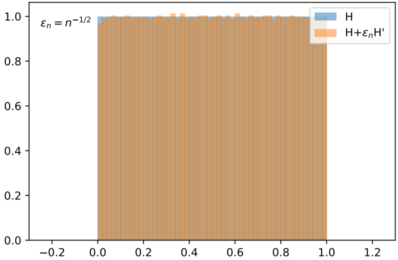
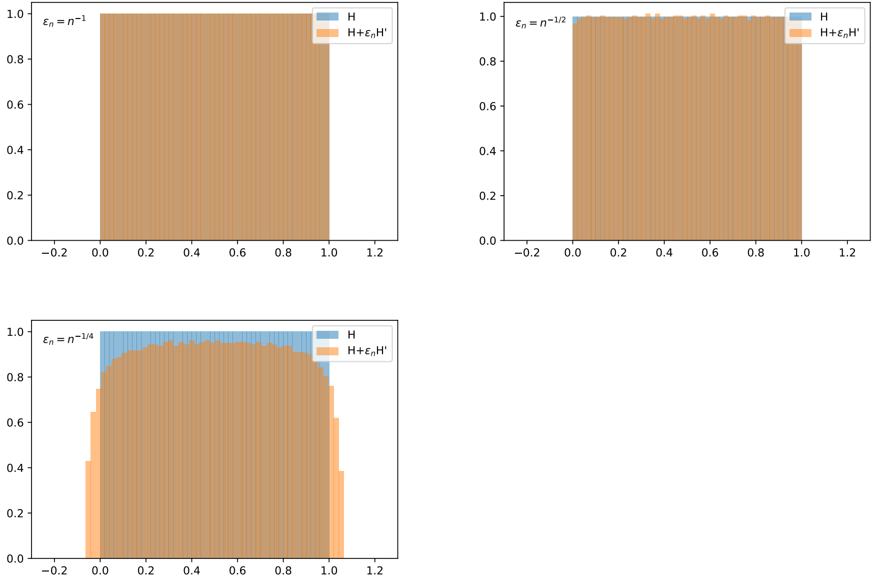
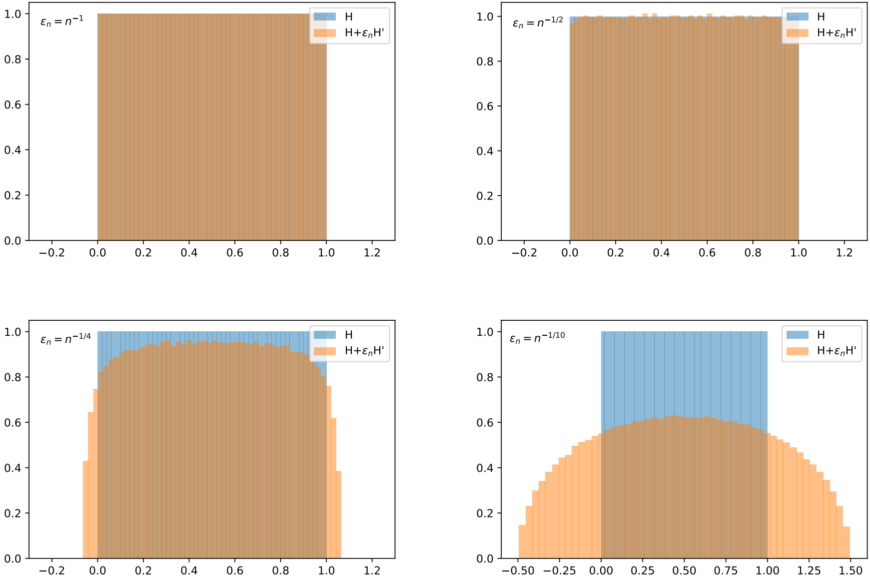


$\varepsilon_n = n^{-1/2}$
H
H+ε_nH'



$\varepsilon_n = n^{-1}$
H
H+ε_nH'
$\varepsilon_n = n^{-1/2}$
H
H+ε_nH'
$\varepsilon_n = n^{-1/4}$
H
H+ε_nH'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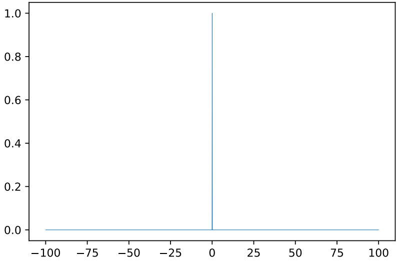
H+ε_nH'
$\varepsilon_n = n^{-1/2}$
H
H+ε_nH'
$\varepsilon_n = n^{-1/4}$
H
H+ε_nH'
$\varepsilon_n = n^{-1/10}$
H
H+ε_nH'

Rewriting of the problem

As any Hermitian matrix can be diagonalized by a unitary matrix, U, we can rewrite this problem as :

$$\underbrace{UH_nU^*}_{D_n} + \varepsilon_n \cdot \underbrace{UH'_nU^*}_{X_n}$$

where D_n is a diagonal matrix and X_n an hermitian matrix.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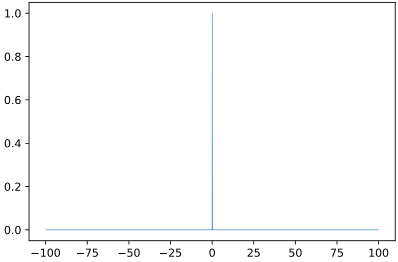
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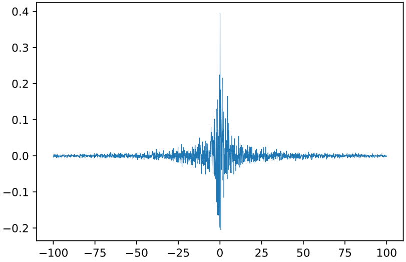
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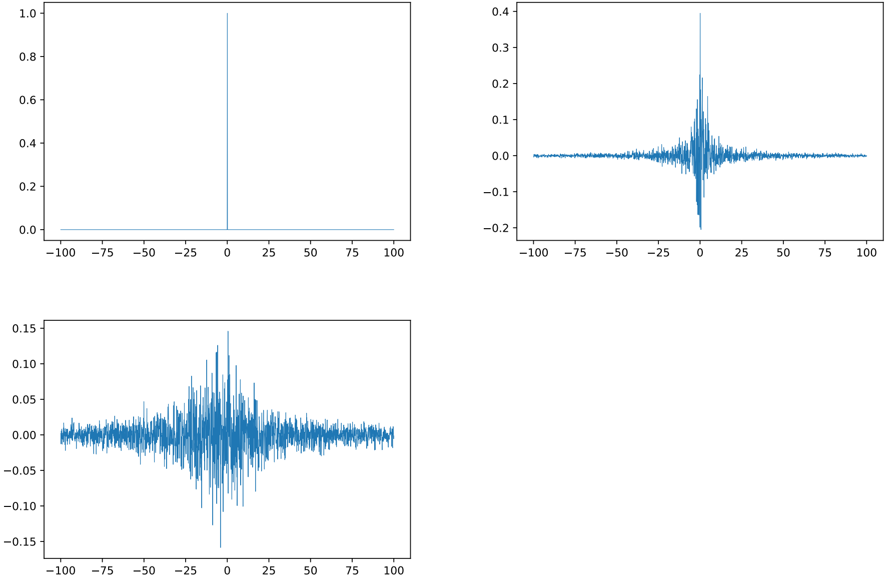
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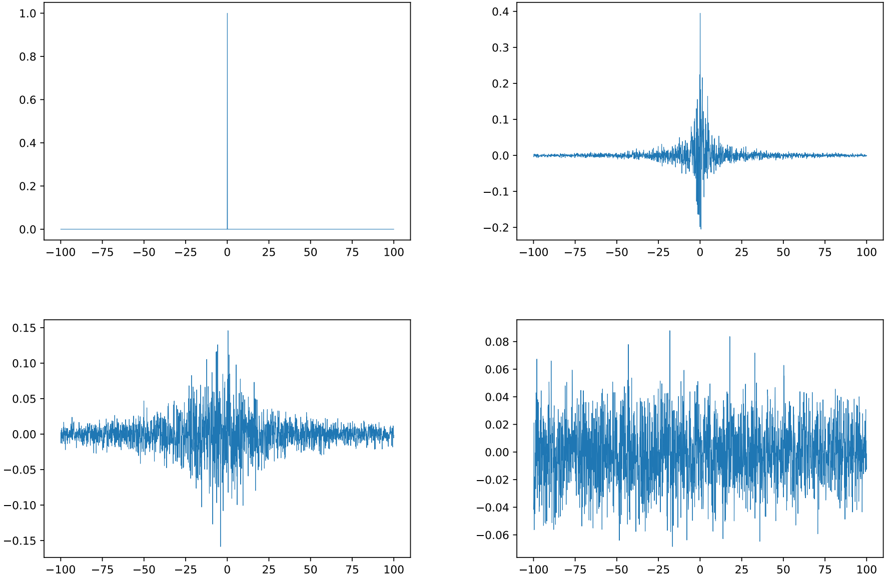

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0.02
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Study of the eigenvalues

Study of the spectrum of a perturbed matrix

$$D_n^\varepsilon := D_n + \varepsilon_n . X_n$$

Let denotes

- μ_n^ε the empirical spectral distribution of D_n^ε
- μ_n the empirical spectral distribution of D_n

Our aim is to give a pertubative expansion of μ_n^ε around μ_n.

Theorem (F.Benaych-Georges, N.Enriquez and A.Michaïl)

For all compactly supported $\mathcal{C}^6$ function on $\mathbb{R}$, the following convergences hold:

- **Perturbative regime:** *if $\varepsilon_n \ll n^{-1}$, then,*

$$n\varepsilon_n^{-1}(\mu_n^{\varepsilon} - \mu_n)(\phi) \quad \xrightarrow[n\to\infty]{\text{dist.}} \quad Z_{\phi}.$$

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- **Critical regime:** *if $\varepsilon_n \sim c/n$, with c constant, then,*

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- **Semi-perturbative regime:** *if $n^{-1} \ll \varepsilon_n \ll n^{-1/3}$, then,*

$$n\varepsilon_n^{-1}\left((\mu_n^\varepsilon - \mu_n)(\phi) + \varepsilon_n^2\int \phi'(s)F(s)\mathrm{d}s\right) \quad \xrightarrow[n\to\infty]{\text{dist.}} \quad Z_\phi .$$

Random term of the expansion

The random term of the expansion is a random field, $(Z_\phi)_{\phi \in \mathcal{C}^6}$, indexed by the space of complex $\mathcal{C}^6$ functions on $\mathbb{R}$, which can be represented as

$$Z_\phi = \int_0^1 \sigma_d(t)\phi'(f(t))\mathrm{d}B_t$$

where, (B_t) is the standard one-dimensional Brownian motion.

$\rightarrow$ σ_d and f are limit parameters of the diagonal entries of X_n and D_n

Idea of the proof

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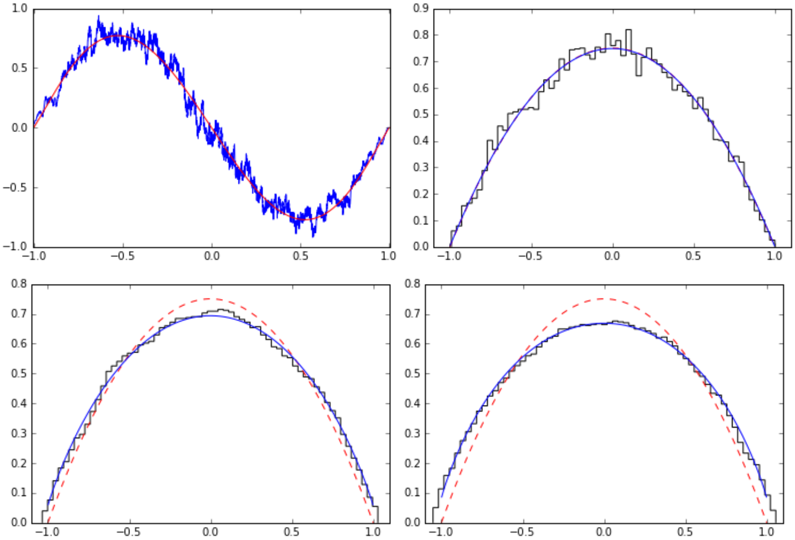
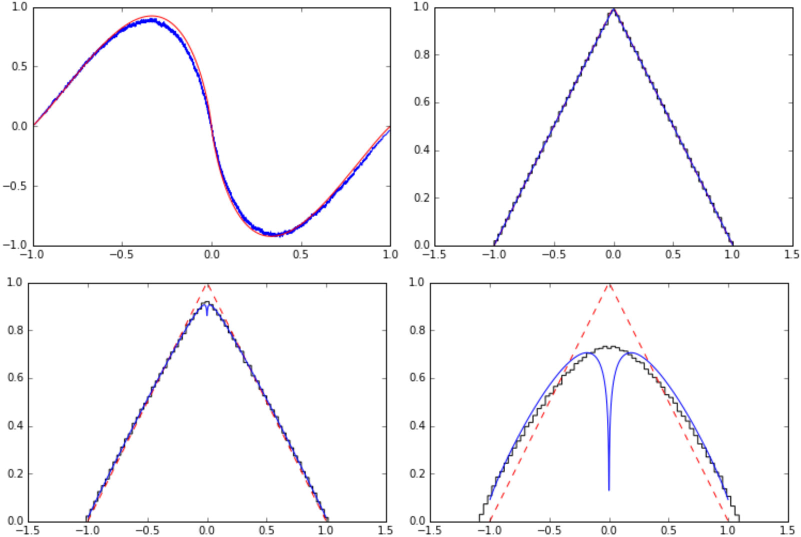
Perturbation of the parabolic pulse distribution by a GOE matrix



Perturbation of the triangular pulse distribution by a GOE matrix



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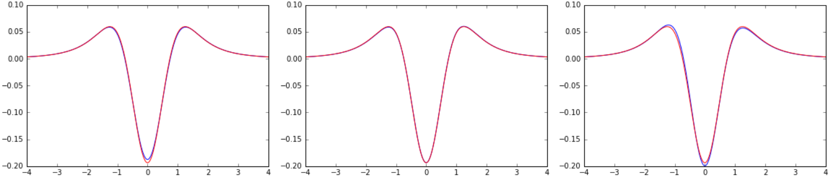

Study of the eigenvectors

Let us denote $\lambda_1, \dots, \lambda_n$ and $\vec{u_1}, \dots, \vec{u_n}$ the eigenvalues and the associated eigenvectors of a $n \times n$ Hermitian matrix H_n.

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Definition (Empirical spectral measure)

The empirical spectral measure of H_n is the probability measure defined by:

$$\mu_{H_n} = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}$$

Definition (Spectral measure)

The spectral measure of H_n over a vector $\vec{v}$, $\mu_{H,\vec{v}}$, is the probability measure defined by:

$$\mu_{H,\vec{v}} = \sum_{j=1}^{n} |\vec{v} \cdot \vec{u_j}|^2 \delta_{\lambda_j}$$

Remark

Note that for a $n \times n$ Hermitian matrix, H_n, with eigenvalues $\lambda_1, \ldots, \lambda_n$ and associated eigenvectors $\vec{u}_1, \ldots, \vec{u}_n$,

$$(\phi(H_n))_{i,i} = \sum_{j=1}^{n} |\vec{e}_i \cdot \vec{u}_j|^2 \phi(\lambda_j)$$

Here we focus on the spectral measure $\mu^{\varepsilon}_{n,\mathsf{e}_i}$ of D^{ε}_n over a vector e_i of the canonical basis, defined through the eigenvector basis $(\mathsf{u}^{\varepsilon}_j)_{j\in\{1,\dots,n\}}$ and their related eigenvalues $(\lambda^{\varepsilon}_j)_{j\in\{1,\dots,n\}}$ of D^{ε}_n by

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Property (Key identity)

$$\mu^{\varepsilon}_{n,\mathrm{e}_i}(\phi) = \int \phi(x)\, \mathrm{d}\mu^{\varepsilon}_{n,\mathrm{e}_i}(x) = \sum_{j=1}^{n} \left|\langle \mathrm{u}^{\varepsilon}_j, \mathrm{e}_i\rangle\right|^2 \phi(\lambda^{\varepsilon}_j) = \left(\phi\left(D^{\varepsilon}_n\right)\right)_{i,i}$$

Main result

Theorem

Let us suppose that $\varepsilon \ll 1$. Let $\varphi : \mathbb{R} \to \mathbb{C}$ be a compactly supported $\mathcal{C}^5$ function. For $x \in [0,1]$, set $i = i(n,x) = \lfloor nx \rfloor$. Then,

$$\int_{\mathbb{R}} \varphi(t) \mathrm{d}\mu^{\varepsilon}_{n,\mathrm{e}_i}(t) = \varphi\left(\lambda_i + \frac{\varepsilon}{\sqrt{n}} x_{ii}\right) + \varepsilon^2 \int_{\mathbb{R}} \varphi''(z) \zeta_{f(x)}(z) \mathrm{d}z + O_{L^2}\left(\varepsilon^2 {\varphi^{(5)}}_{\infty} (\varepsilon + \frac{1}{\sqrt{n}} + \eta_n)\right)$$

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$$+ O_{L^2}\left(\varepsilon^2 {\varphi^{(5)}}_{\infty}(\varepsilon + \frac{1}{\sqrt{n}} + \eta_n)\right)$$

where, for any $s \in \mathbb{R}$, the function ζ_s is defined on $\mathbb{R}$ by

$$\zeta_s(z) := \int_1^{+\infty} \frac{r-1}{r^2} \tau(s, s + r(z-s)) \rho(s + r(z-s)) \mathrm{d}r.$$

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2. Then, we extend this convergence to the larger class of compactly supported $\mathcal{C}^5$ functions on $\mathbb{R}$, thanks to the Helffer–Sjöstrand formula and a Lemma of Shcherbina and Tirozzi.

Consquence for the eigenvectors

$$\mu_n^\varepsilon := \frac{1}{n}\sum_{j=1}^{n}\delta_{\lambda_j^\varepsilon} \qquad ; \qquad \mu_{n,\mathrm{e}_i}^\varepsilon = \sum_{j=1}^{n}|\langle \mathrm{u}_j^\varepsilon, \mathrm{e}_i\rangle|^2\delta_{\lambda_j^\varepsilon}$$

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The definition of $\mu_{n,\mathrm{e}_i}^\varepsilon$ implies, for $t \neq f(x)$,

$$\mu_{n,\mathrm{e}_i}^\varepsilon(\mathrm{d}t) \approx n|\langle \mathrm{u}_j^\varepsilon, \mathrm{e}_i \rangle|^2 \mu_n^\varepsilon(\mathrm{d}t).$$

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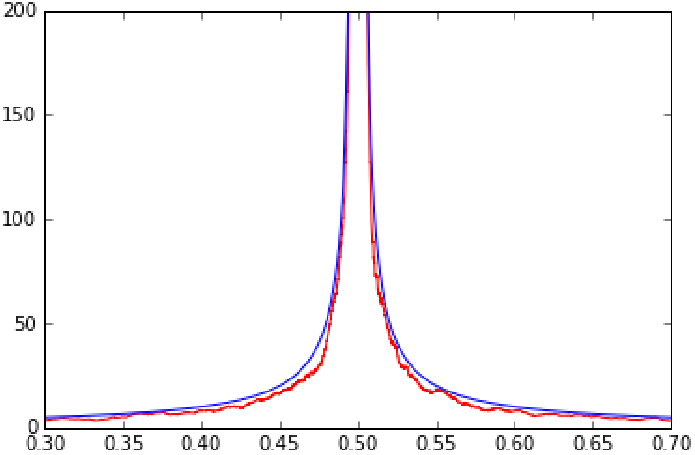
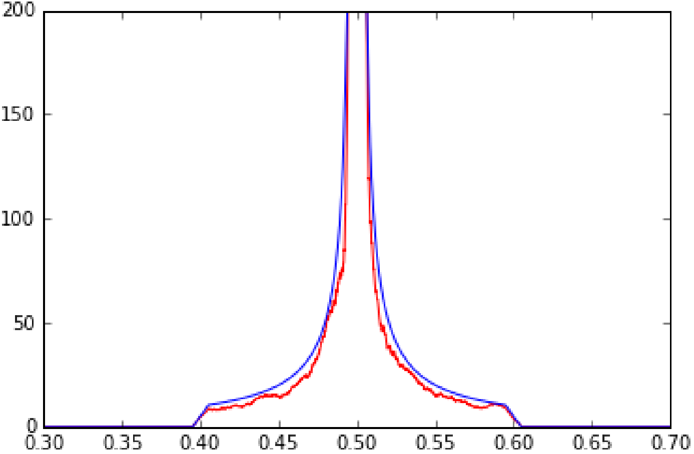
Thus in the sense of the distribution,

$$\varepsilon^{-2} n|\langle \mathrm{u}^\varepsilon_{\lfloor nf^{-1}(t)\rfloor}, \mathrm{e}_{\lfloor nx\rfloor}\rangle|^2 \ \approx \ \frac{\tau(f(x),t)}{(t-f(x))^2}.$$

Perturbation by a GOE matrix



Perturbation by a Gaussian band matrix



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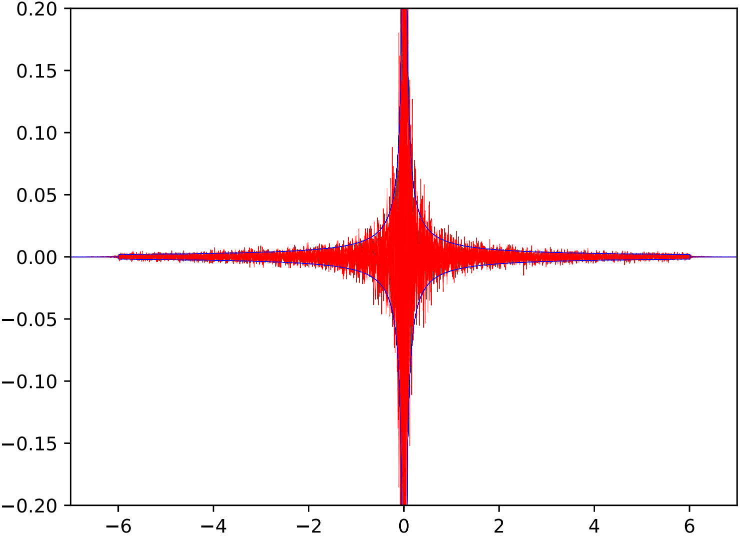

Merci